JOURNAL OF APPROXIMATION THEORY **85**, 27–42 (1996)
ARTICLE NO. 0027

Codimension One Minimal Projections Onto the Quadratics

MICHAEL PROPHET

Department of Mathematics, Idaho State University, Pocatello, Idaho 83209

Communicated by E. W. Cheney

Received August 13, 1993; accepted in revised form February 22, 1995

$V_3=[1, t, t_2]$, for all $\sigma \geqslant 1$. This generalizes a result of G. J. O. Jameson. © 1996 Academic Press, Inc.

1. INTRODUCTION AND PRELIMINARIES

A 1987 paper of G. J. O. Jameson (see [3]) established a lower bound for the projection constant for the second degree algebraic polynomials on $[-1, 1]$, Π_2. The method used was to consider overspaces of Π_2 of the form $X=[1, t, t_2, t\,|t|^{\sigma}]$ for $\sigma=1, 2$ and establish lower bounds for projections from X onto Π_2. This would provide lower bounds for a minimal projection from $C[-1, 1]$ onto Π_2. Good good estimates were attained by cleverly chosing certain 'extreme' families of function from X to project. In 1990, the projection constant for Π_2 was found by B. L. Chalmers and F. T. Metcalf (see [1]). In this paper we extend Jameson's results by describing a procedure to find a minimal projection from $X=[1, t, t_2, t\,|t|^{\sigma}] \to \Pi_2$ for all $\sigma \geqslant 1$.

A subspace $Y \subset C[-1, 1]$ is said to be symmetric if $f \in Y$ implies $f^* \in Y$, where $f^*(t)=f(-t)$. An operator P: $Y \to V$ (Y, V symmetric subspaces) is said to be symmetric if $Pf^*=(Pf)^*$ for all $f \in Y$. When searching for a minimal projection it suffices to consider only symmetric projections since any projection P can by symmetrized by defining $\hat{P}f=\frac{1}{2}((Pf^*)^*+Pf)$. This gives $\|\hat{P}\| \leqslant \|P\|$. P is symmetric if and only if P takes even/odd functions to even/odd functions.

0021-9045/96 $18.00

The symmetric projections from $X \to \Pi_2$ form a one-parameter family of operators, since each projection is uniquely determined by where $t\,|t|^{\sigma}$ is sent and, since this function is odd, we must have $Pt\,|t|^{\sigma} = \alpha t$ for some α. Thus we write P_α for a symmetric projection.

For $\sigma \geqslant 1$, we establish an analog of the third degree Chebyshev polynomial. Define $\hat{T}_{\sigma+1}(t) = (1/\sigma\beta_0^{\sigma+1})\, t\,|t|^{\sigma} - ((\sigma+1)/\sigma\beta_0))\, t$, where β_0 is the unique solution to $H(\beta) = \sigma\beta^{\sigma+1} + (\sigma+1)\,\beta^{\sigma} - 1 = 0$ on $[0, 1]$. Then $\hat{T}_{\sigma+1}(t)$ is a norm 1 odd function with $\hat{T}_{\sigma+1}(\beta_0) = -1$, $\hat{T}'_{\sigma+1}(\beta_0) = 0$ and $\hat{T}_{\sigma+1}(1) = 1$ (and corresponding values at $t = -\beta_0$ and $t = -1$). Denoting the monic version of a function $f(t)$ by $m(f(t))$, one can easily check that $t\,|t|^{\sigma}$ is uniquely best approximated from Π_2 by $t\,|t|_{\sigma} - m(\hat{T}_{\sigma+1}(t))$.

We will need the following result from topological degree theory.

LEMMA 1.1. *Let $F\colon D \subset R^2 \to R^2$ be continuous in the simply connected domain D. Let $G \subset D$ be a domain with boundary $\tau(t)$, a simple closed curve. If the winding number of the image of τ under F with respect to the origin is not zero (i.e. $\omega(F\tau, 0) \neq 0$) then there exists $z \in G$ such that $F(z) = 0$.*

Proof. We will show the contrapositive in the complex plane. Fix $z_0 \in \tau$. Then z_0 is homotopic to τ. By the continuity F, we also have Fz_0 homotopic to $F\tau$ in $\mathscr{C} - \{0\}$. Since $1/z$ is analytic in $\mathscr{C} - \{0\}$, we have

$$\omega(F\tau, 0) = \frac{1}{2\pi i} \int_{F\tau} \frac{1}{z}\, dz = \frac{1}{2\pi i} \int_{Fz_0} \frac{1}{z}\, dz = 0. \quad \blacksquare$$

We now give a characterization for minimal projections on finite dimensional spaces (see [2] for proof). Let X be a real finite dimensional normed space and V an n-dimensional subspace. Let $S(X)$ and $B(X)$ denote the unit sphere and unit ball, respectively, Let $\mathscr{B} = \mathscr{B}(X, V)$ be the space of all bounded linear operators from X to V and $\mathscr{P}$ be the subset of all projections.

DEFINITION 1.1. For $P \in \mathscr{P}$ define the set of extremal pairs of P as $\mathscr{E}(P) = \{(x, y) \in S(X) \times S(X^*) \mid \langle Px, y\rangle = \|P\|\}$.

Notation. For $u \in X^*$, $v \in X$ define $u \otimes v\colon X \to X$ by $\langle x, u \otimes v\rangle = \langle x, u\rangle\, v$. Thus each pair $(x, y) \in \mathscr{E}(P)$ can be associated with the operator $y \otimes x$.

THEOREM 1.1. *$P \in \mathscr{P}$ has minimal norm if and only if there exists an operator $E_P \in \overline{co}\{\mathscr{E}(P)\}$ such that V is an invariant subspace of E_P.*

2. Main Results

THEOREM 2.1. *For $\sigma \geqslant 1$, the minimal projection from X onto V_3 is given by $P_\alpha = \sum_{i=1}^{3} (u_i \otimes v_i)$ where $v_i(t) = t^{i-1}$ and*

$$\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ \dfrac{-1}{2\alpha^{1/\sigma}} & 0 & \dfrac{1}{2\alpha^{1/\sigma}} \\ \dfrac{1}{2\alpha} & \dfrac{-1}{\alpha} & \dfrac{1}{2\alpha} \end{pmatrix} \begin{pmatrix} \delta_{-\alpha^{1/\sigma}} \\ \delta_0 \\ \delta_{\alpha^{1/\sigma}} \end{pmatrix}$$

where $\alpha \in [(\beta_0)^\sigma, 1]$ and δ_t denotes point evaluation at t.

The norms of some of these minimal projections are given at the this paper. We prove the following lemmas in order to establish the above.

LEMMA 2.1. *For $-1 \leqslant t_1 < 0 \leqslant t_2 < 1$, we have $X^* = [\delta_1, \delta_{t_2}, \delta_{t_1}, \delta'_{t_1}]$.*

LEMMA 2.2. *For $t \in (-1, 1)$, we have $X^* = [\delta_1, \delta_t, \delta'_t, \delta_{-1}]$.*

The proofs of these lemmas are omitted since they simply involve verifying non-zero determinants. In the above, δ'_t denotes first derivative evaluation at t. From Lemma 2.1 we have the following definition.

DEFINITION 2.1. Let η, $\beta \in [0, 1]$ with $\eta \leqslant \beta$ and $\eta \neq 1$. Then $f_{\eta,\beta}(t) = At\,|t|^\sigma + Bt^2 + Ct + D \in X$ is the unique function satisfying $f_{\eta,\beta}(1) = 1$, $f_{\eta,\beta}(\eta) = -1$, $f_{\eta,\beta}(-\beta) = 1$, $f'_{\eta,\beta}(\beta) = 0$.

Note 1. The coefficiednts of $f_{\eta,\beta}(t) = At\,|t|_\sigma + Bt^2 + Ct + D$ are found by solving a linear system. They are given by

$$\begin{pmatrix} A \\ B \\ C \\ D \end{pmatrix} = \frac{-2}{K} \begin{pmatrix} 1 \\ F'(\beta) \\ (\beta - 1)\,F'(\beta) - F(\beta) \\ F(\beta) - \beta F'(\beta) - 1 - K/2 \end{pmatrix} \tag{1}$$

where

$$F(\beta) = \frac{1 + \beta^{\sigma+1}}{1 + \beta}, \qquad F'(\beta) = \frac{dF}{d\beta}$$

and

$$K = (\eta - 1)\,[F'(\beta)(\beta + \eta) - F(\beta)] + \eta^{\sigma+1} - 1.$$

DEFINITION 2.2. $\mathscr{F}_{\eta,\beta} = \{ f_{\eta,\beta}(t) \in X \mid G_1(\eta, \beta) = 0\}$, where $G_1(\eta, \beta) = f'_{\eta,\beta}(\eta)$.

We show $\mathscr{F}_{\eta,\beta}$ is non-empty in Lemma 2.5.

LEMMA 2.3. *If $f(t) = At\,|t|^{\sigma} + Bt^2 + Ct + D \in \mathscr{F}_{\beta}$ (i.e. $G_1(\beta, \eta) = 0$) and $\beta \geqslant \beta_0$, $\eta \leqslant \beta$ then $\| f \| = 1$.*

Proof. For $f \in \mathscr{F}_{\beta}$, we have $f(1) = f(-\beta) = 1$, $f(\eta) = -1$ and $f'(-\beta) = f'(\eta) = 0$. Also note for any $f \in X$, $f''(t) = A\sigma(\sigma+1)\, sgn(t)\, |t|^{\sigma-1}$. Thus f'' has at most one zero, so f' has at most 2 zeroes. It follows that if $f \in \mathscr{F}_{\beta}$ then $f(\eta)$ is a relative minimum and $f(-\beta)$ a relative maximum. It is clear that if $|f(t_0)| > 1$ for $t_0 \in [-\beta, 1]$, another relative extreme point would be necessary and this would imply that f' has more than 2 zeroes. So $|f(t)| \leqslant 1$ on $[-\beta, 1]$. We now show $|f(-1)| \leqslant 1$ to conclude $\| f \| = 1$. Since $f(-1) \leqslant 1$ is clear, we show $f(-1) \geqslant -1$.

First note from the definition of the coefficients that we have

$$f(-1) = -A + B - C + D = -2(A + C) + 1.$$

We show $f(-1)$ to be a continuous function of $\beta \in [\beta_0, 1]$. Recall

$$G_1(\beta, \eta) = \frac{-2[(\sigma+1)\,\eta^{\sigma} + 2F'(\beta)\,\eta - (1-\beta)\,F'(\beta) - F(\beta)]}{K}, \tag{2}$$

and

$$G_1(\beta, \eta) = 0 \Leftrightarrow G_{\beta}(\eta) = (\sigma+1)\,\eta^{\sigma} - F(\beta) + F'(\beta)(2\eta + \beta - 1) = 0. \tag{3}$$

Fix $\beta \in [\beta_0, 1]$. Then

$$G_{\beta}(0) = -F(\beta) + F'(\beta)(\beta - 1) < 0$$

since $F'(\beta) = H(\beta)/(1+\beta)^2$. Also

$$G_{\beta}(\beta) = \frac{H(\beta)}{(1+\beta)^2}\,(4\beta) \geqslant 0$$

with equality only when $\beta = \beta_0$. Thus for each $\beta \in [\beta_0, 1]$, $G_{\beta}(\eta) = 0$ for some $\eta \in [0, \beta]$. Furthermore since

$$\frac{dG_{\beta}}{d\eta} > 0$$

the zero in $[0, \beta]$ is unique and varies continuously with β. Therefore, we write $\eta = \eta(\beta)$ as the continuous function of β which yields the solution to $G_{\beta}(x) = 0$ on $[0, \beta]$. Thus $f(-1)$ is a continuous function of $\beta \in [\beta_0, 1]$.

Now note for $\beta=1$, $f(-1)=1$ by the definition of $f\in\mathscr{F}_{\eta,\beta}$. And $f(-1)=-1$ for $\beta=\beta_0$ ($\beta=\beta_0$ corresponds to $\hat{T}_{\sigma+1}$). Suppose for some $\beta\in[\beta_0,1]$ we have $f_\beta(-1)<-1$. Then there exists $\beta_*\in(\beta_0,1)$ such that $f_{u_*}(-1)=-1$. So we can conclude $\|f_{u_*}\|=1$. Furthermore, since $f_{u_*}\in\mathscr{F}_{\eta,\beta}$, we have $f_{u_*}(-1)=f_{u_*}(\eta)=-1$ and $f_{u_*}(-\beta_0)=f_{\beta_0}(1)=1$. With $m(f_{u_*}(t))$ as the monic version of $f_{u_*}(t)$ we would have $t\,|t|^\sigma-m(f_{\beta_0}(t))$ as a best approximate to $t\,|t|^\sigma$; but this contradicts the fact that $\hat{T}_{\sigma+1}$ is the best approximate. Thus $f(-1)\geqslant -1$ and $\|f\|=1$. ▮

Lemma 2.2 allows us to define the following.

Definition 2.3. Let $\gamma\in(-1,0)$. Then $f_\gamma(t)=at\,|t|^\sigma+bt^2+ct+d\in X$ is the unique function satisfying $f_\gamma(1)=-1$, $f_\gamma(\gamma)=1$, $f'_\gamma(\gamma)=0$, $f_\gamma(-1)=1$.

Note 2. The coefficients of f_γ are given by

$$d=\frac{\sigma\,|\gamma|^{\sigma+1}-(\sigma+1)\,|\gamma|^\sigma+1}{(\sigma-1)\,|\gamma|^{\sigma+2}-(\sigma+1)\,|\gamma|^\sigma+\gamma^2+1}$$

and

$$c=\frac{(\gamma^2-1)\,D-|\gamma|^{\sigma+1}+1}{|\gamma|^{\sigma+1}+\gamma},\qquad b=-d,\ a=-(1+c).$$

Definition 2.4. $\mathscr{F}_\gamma=\{f_\gamma\in X\mid\gamma\in(-1,0)\}$.

Lemma 2.4. *If $f(t)\in\mathscr{F}_\gamma$, then $\|f\|=1$.*

Proof. For $f(t)=at\,|t|^\sigma+bt^2+ct+d\in\mathscr{F}_\gamma$ we have $f(-1)=f(\gamma)=1$, $f(1)=-1$ and $f'(\gamma)=0$ (and recall, for any $f\in X$, f' has at most two zeroes). Note f' must have a zero in $(-1,\gamma)$. It follows that $f(\gamma)$ is a maximum and thus $|f(t)|\leqslant 1$ for $t\in[\gamma,1]$. Clearly $f(t)\leqslant 1$ on $[-1,\gamma]$. We now show that $f(t)\geqslant 0$ for $t\leqslant 0$. Since f' has all its zeroes in $[-1,1]$, we can conclude $f(t)\to\infty$ as $t\to-\infty$. Thus the coefficient $a<0$. This says that f'' is decreasing and we know $f''(\gamma)<0$. Since $f(\gamma)=1$, $f'(\gamma)=0$, and $f(1)=-1$ we must have $f(0)=d>0$. Since $d<1$ we have $c<0$. Now for $t<0$, $f(t)>0$ follows from the signs of the coefficients. ▮

Definition 2.5. Fix $\sigma\geqslant 1$, $\alpha\in[\beta_0^\sigma,1]$ and $\rho\in[-1,0]$. Then define the functional $L_\rho=\delta_\rho\circ P_\alpha\in X^*$.

Note 3. $\|P_\alpha\|=\max_{\rho\in[-1,0]}\|L_\rho\|$. Note that $\|L_0\|=\|L_{-\alpha^{1/\sigma}}\|=1$.

Definition 2.6. Let $\phi\in X^*$. If there exists $\{t_i\}_{i=1}^n\subset[-1,1]$ and constants $\{c_i\}_{i=1}^n$ such that $\phi=\sum_{i=1}^n c_i\delta_i$ with $\|\phi\|=\sum_{i=1}^n|c_i|$, then we say this representation of ϕ is a canonical representation.

LEMMA 2.5. *Fix* $\alpha \in [(\beta_0)^\sigma, 1]$ *and* $\rho \in [-1, -\alpha^{1/\sigma}]$. *Then there exists constants* $\{c_i\}_{i=1}^3$ *and* β, $\eta \in [0, 1]$ *with* $\beta \geqslant \alpha^{1/\sigma}$ *and* $\eta \leqslant \beta_0$ *such that the representation* $L_\rho = c_1\delta_1 + c_2\delta_\eta + c_3\delta_{-\beta}$ *is a canonical representation. Furthermore,* $f_{\eta,\beta} \in \mathscr{F}_{\eta,\beta}$ *is an extremal for* L_ρ.

Proof. A representation of L_ρ must agree with L_ρ on the basis for X. Forcing the above representation and L_ρ to agree on $\{1, t, t^2\}$ gives

$$\begin{pmatrix} c_1 \\ \\ c_2 \\ \\ c_3 \end{pmatrix} = \begin{pmatrix} \dfrac{(\rho+\beta)(\rho-\eta)}{(\beta+1)(1-\eta)} \\ \dfrac{(\rho+\beta)(1-\rho)}{(\beta+\eta)(1-\eta)} \\ \dfrac{(\rho-\eta)(\rho-1)}{(\beta+1)(\beta+\eta)} \end{pmatrix}. \tag{4}$$

To ensure agreement on $t\,|t|^\sigma$, we define

$$G_2(\beta, \eta) = c_1 + c_2\eta^{\sigma+1} - c_3\beta^{\sigma+1} - \alpha\rho$$

for c_1, c_2, c_3 above. Thus L_ρ has the above representation if and only if $G_2(\beta, \eta) = 0$ for some $\beta, \eta \in [0, 1]$. Note for $\rho = -\alpha^{1/\sigma}$, L_ρ is a point evaluation and the representation is immediate. This simple representation also occurs in the case $\alpha = 1$, since we must choose $\rho = -1$. Then $L_{-1} = \delta_{-1}$ and again the representation is trivial. Thus, in the following we assume $\alpha < 1$ and $\rho < -\alpha^{1/\sigma}$. Define $G = (G_1, G_2)\colon R^2 \to R^2$ for G_1 and G_2 defined above. Recall the forms of G_1 and G_2:

$$G_1(x, y) = \frac{-2[(\sigma+1)\,y^\sigma + 2F'(x)\,y - (1-x)\,F'(x) - F(x)]}{(y-1)(F'(x)(x+y) - F(x)) + y^{\sigma+1} - 1}$$

$$G_2(x, y) = \frac{(\rho+x)(\rho-y)}{(x+1)(1-y)} + y^{\sigma+1}\frac{(\rho+x)(1-\rho)}{(x+y)(1-y)} - x^{\sigma+1}\frac{(\rho-y)(\rho-1)}{(x+1)(x+y)} - \alpha\rho.$$

We now find a zero of G in the following region. Define $\Omega_\alpha \subset R^2$ as the region bounded by the following four line segments:

$$l_1 = \{(x, 0) \mid \alpha^{1/\sigma} \leqslant x \leqslant -\rho\}$$
$$l_2 = \{(-\rho, y) \mid 0 \leqslant y \leqslant -\rho\}$$
$$l_3 = \{(x, x) \mid \alpha^{\sigma/1} \leqslant x \leqslant -\rho\}$$
$$l_4 = \{(\beta_0, y) \mid 0 \leqslant y \leqslant \alpha^{1/\sigma}\}.$$

Now for $\rho > -1$, the denominators of G_1 and G_2 are never zero in Ω_α and thus G is continuous on Ω_α. We will now prove the lemma first for $\rho > -1$

and consider $\rho = -1$ separately. We will show the image of $\partial\Omega_\alpha$ under G winds around the origin abnd conclude, from Lemma 1.1, that a zero of G exists in Ω_α. We first consider $G(l_1)$. It is easy to check that

$$G_1(x, 0) = 2\left[\frac{(1-x)\,F'(x) + F(x)}{-xF'(x) + F(x) - 1}\right] < 0$$

for $\alpha^{1/\sigma} \leqslant x \leqslant -\rho$. Thus we conclude $G(l_1)$ is a curve staying to the left of the origin (in the (G_1, G_2) plane). Now consider $G(l_2)$. From above we see $G_1(-\rho, 0) < 0$. We claim that $G_1(-\rho, -\rho) > 0$. This will be shown when looking at the image of l_3. Furthermore, observe

$$G_2(-\rho, y) = |\rho|\,(\alpha - |\rho|^\sigma) < 0$$

since $|\rho|^\sigma > \alpha$. Thus $G(l_2)$ is a curve lying below the origin. Consider $G(l_3)$. After much simplification one finds

$$G_1(x, x) = \frac{-4H(x)}{x(\sigma x^{\sigma+1} + 2x^\sigma - \sigma x^{\sigma-1} - 2)}$$

and

$$\sigma x^{\sigma+1} + 2x^\sigma - \sigma x^{\sigma-1} - 2 = 2(x^\sigma - 1) + \sigma x^{\sigma-1}(x^2 - 1) < 0.$$

Since $H(x) > 0$ for $x \in [\alpha^{1/\sigma} - \rho]$ we conclude $G_1(x, x) > 0$. Now consider $G_2(x, x)$. From above, we see that $G_2(-\rho, -\rho) < 0$. Furthermore, one easily finds $G_2(\alpha^{1/\sigma}, \alpha^{1/\sigma}) > 0$. $G(l_3)$ lies to the right of the origin. Finally, we consider $G(l_4)$. With $G_1(\alpha^{1/\sigma}, \alpha^{1/\sigma}) > 0$ and $G_1(\alpha^{1/\sigma}, 0) < 0$ we show $G(l_4)$ lies above the origin by showing $G_2(\alpha^{1/\sigma}, y) > 0$ for $y \in [0\ \alpha^{1/\sigma}]$:

$$G_2(\alpha^{1/\sigma}, y) = \frac{(\rho + \alpha^{1/\sigma})(\rho - y)}{(\alpha^{1/\sigma} + 1)(1 - y)} - y^{\sigma+1}\frac{(\rho + \alpha^{1/\sigma})(\rho - 1)}{(\alpha^{1/\sigma} + y)()1 - y)}$$
$$- \alpha^{(\sigma+1)/\sigma}\frac{(\rho - 1)(\rho - y)}{(\alpha^{1/\sigma} + 1)(\alpha^{1/\sigma} + y)} - \alpha\rho.$$

A common denominator of $(1-y)(\alpha^{1/\sigma} + 1)(\alpha^{1/\sigma} + y) > 0$ can be used to combine the above. Using the inequality $\rho < -\alpha^{1/\sigma}$, one finds the numerator to be positive and $G_2(\alpha^{1/\sigma}, y) > 0$ for $y \in [0, \alpha^{1/\sigma}]$. This demonstrates that the image of $\partial\Omega_\alpha$ under G has a nonzero winding number with respect to the origin. By Lemma 1.1 we have a zero of G in Ω_α for the case $\rho \in (-1, -\alpha^{1/\sigma})$. For $\rho = -1$, define

$$\hat{G} = (\hat{G}_1, \hat{G}_2) = \left(\frac{G_1}{1 + |G_1|}, \frac{G_2}{1 + |G_2|}\right).$$

Note $\hat{G}$ is continuous on Ω_α and the zeroes of $\hat{G}$ and G coincide. Furthermore, since $\operatorname{sgn}(\hat{G}_i) = \operatorname{sgn}(G_i)$, it is clear from the examination of $G(\partial\Omega)$ that the winding number of $\hat{G}(\partial\Omega_\alpha)$ with respect to the origin is also non-zero. Now for $\alpha \in [(\beta_0)^\sigma, 1]$ and $\rho \in [-1, -\alpha^{1/\sigma}]$ we have $L_\rho = c_1\delta_1 + c_2\delta_\eta + c_3\delta_{-\beta}$ for β, $\eta \in \Omega_\alpha$ and c_i as in (4). To see this representation is canonical, observe from (4) that $c_1 \geqslant 0$, $c_2 \leqslant 0$ and $c_3 \geqslant 0$ (this follows immediately from β, $\eta \in \Omega_\alpha$). Furthermore, since $G_1(\beta\,\eta) = 0$ we have that $f_{\beta\eta} \in \mathscr{F}_{\eta\beta}$. Therefore

$$L_\rho f_{\beta\eta} = c_1 - c_2 + c_3 = |c_1| + |c_2| + |c_3| = \|L_\rho\|.$$

Thus, the representation is canonical and $f_{\beta,\eta}$ is an extremal for L_ρ ∎

LEMMA 2.6. *The canonical representation for L_ρ given in Lemma* 2.5 *is unique* (*or, equivalently, G has a unique zero in Ω_α*).

Proof. Uniqueness of canonical representations of functionals on polynomial spaces is given in [4]. The result easily generalizes to our $X = [1, t, t^2, t\,|t|^\sigma]$ with $\sigma > 1$. For $\sigma = 1$ we can solve for η in terms of β when $G_1(\beta, \eta) = 0$; one finds $\eta = 1/(2+\beta)$. Also consider $G_2(\beta\,\eta)$ for $\sigma = 1$ and $\rho = -1$ (we consider $\rho = -1$ since this will eventually be the only ρ of interest)

$$G_2(\beta\,\eta) = \frac{(\alpha-3)\,\beta^3 + 3(\alpha-3)\,\beta^2 + 3(\alpha+1)\,\beta + \alpha + 1}{(1+\beta)^3}.$$

It is easily seen (by checking the derivative of the numerator) that this function has a unique zero on $[0, 1]$. It is also obvious that this zero changes continuously with α. ∎

LEMMA 2.7. *For $\alpha \in [(\beta_0)^\sigma\ 1]$ and $\rho \in [-1\ -\alpha^{1/\sigma}]$ we have $\|L_{-1}\| \geqslant \|L_\rho\|$.*

Proof. Fix ρ and let $f_\rho = f_{\beta\eta} \in \mathscr{F}_\beta$ denote the extremal of L_ρ. Then $L_\rho\| = P_\alpha f_\rho(\rho)$. Writing $f_\rho(t) = At\,|t|^\sigma + Bt^2 + Ct + D$ where the coefficients are as in (1) we have $P_\alpha f_\rho(t) = Bt^2 + (\alpha A + C)\,t + D$ and we claim

$$(P_\alpha f_\rho)'\,(t) = 2Bt + \alpha A + C \leqslant 0\ t \in [-1, 0]. \tag{5}$$

Recalling the formulas for A, B, and C in (1), note that

$$K = (\eta - 1)[F'(\beta)(\beta + \eta) - F(\beta)] + \eta^{\sigma+1} - 1 < 0$$

since

$$F'(\beta)(\beta+\eta) - F(\beta) = \frac{H(\beta)}{(1+\beta)^2} - \frac{1+\beta^{\sigma+1}}{1+\beta} \geqslant -1.$$

It follows that $A, B > 0$ and $C > 0$. Now note $\alpha A + C \geqslant 2Bt + \alpha A + C$ $t \in [-1, 0]$. But

$$\alpha A + C = \frac{-2}{K}((\beta - 1) F'(\beta) + \alpha - F(\beta))$$

and $-2/K > 0$. Also, recalling that $F'(\beta) > 0$ and $\beta > \alpha^{1/\sigma}$ from Lemma 2.5, we have

$$F(\beta) = \frac{1 + \beta^{\sigma+1}}{1 + \beta} \geqslant \frac{1 + \alpha^{(\sigma+1)/\sigma}}{1 + \alpha^{1/\sigma}} \geqslant \alpha.$$

Thus

$$0 \geqslant \alpha - F(\beta) \geqslant (\beta - 1) F'(\beta) + \alpha - F(\beta)$$

and (5) follows. Since $P_\alpha f_\rho(t)$ is decreasing on $[-1, 0]$ we have

$$\|L_{-1}\| = P_\alpha f_{-1}(-1) \geqslant P_\alpha f_\rho(-1) \geqslant P_\alpha f_\rho(\rho) = \|L_\rho\|. \quad \blacksquare$$

LEMMA 2.8. *$\|L_{-1}\|$ is a continuous function of α, where $\alpha \in [(\beta_0)^\sigma, 1]$.*

Proof. From the definition of the coefficients in (4), we have

$$\|L_{-1}\| = \frac{\eta^2 + \beta\eta - \eta + 3\beta - 4}{(\eta - 1)(\eta + \beta)}$$

where β and η are such that $G(\beta, \eta) = 0$ in Ω_α. By the simple dependence of G on α (G_1 is independent of α and α occurs in G_2 as a constant) and by the *uniqueness* of the zero of G in Ω_α, it follows that this zero varies continuously with α (an assumption of a discontinuity leads to an immediate contradiction). Thus β and η vary continuously with α. $\blacksquare$

LEMMA 2.9. *Fix $\alpha \in [(\beta_0)^\sigma\ 1]$ and $\rho \in [-\alpha^{1/\sigma}\ 0]$. Then there exists constants c_i and $\gamma_0 \in (-1, 0]$ such that the representation $L_\rho = c_1\delta_1 + c_2\delta_{\gamma_0} + c_3\delta_{-1}$ is a canonical representation. Furthermore $f_{\gamma_0} \in \mathscr{F}_\gamma$ is an extremal for L_ρ.*

Proof. Obtaining agreement on $\{1, t, t^2\}$ between L_ρ and the above representation gives

$$\begin{pmatrix} c_1 \\ \\ c_2 \\ \\ c_3 \end{pmatrix} = \begin{pmatrix} \dfrac{(\rho+1)(\rho-\gamma)}{2(1-\gamma)} \\ \\ \dfrac{(\rho^2-1)}{(\gamma^2-1)} \\ \\ \dfrac{(\rho-1)(\rho-\gamma)}{2(1+\gamma)} \end{pmatrix}. \tag{6}$$

To force agreement on all of X, we must have

$$\langle t\,|t|^{\sigma}, L_{\rho}\rangle = \alpha\rho = c_1 + c_2\gamma\,|\gamma|^{\sigma} - c_3$$

or equivalently

$$c_1 + c_2\gamma\,|\gamma|^{\sigma} - c_3 - \alpha\rho = \frac{(\rho^2-1)\ \gamma\,|\gamma|^{\sigma} + \rho(1-\alpha)\,\gamma^2 + (1-\rho^2)\,\gamma - \rho + \alpha\rho}{\gamma^2-1} = 0. \tag{7}$$

The numerator in (7) has two zeroes in $[-1, 0]$: $\gamma = -1$ and $\gamma = \gamma_0$ where $\gamma_0 \in [\rho, 0]$. In the case $\alpha = 1$, $\gamma_0 = 0$ for all ρ. In thze case $\rho = -\alpha^{1/\sigma}$, note $\gamma_0 = \rho$. Since $0 \geqslant \gamma_0 \geqslant \rho$, the coefficients in (6) are such that $c_1 \leqslant 0$ and $c_2, c_3 \geqslant 0$. Recalling the properties of $f_{\gamma_0} \in \mathscr{F}_{\gamma}$ we have

$$L_{\rho} f_{\gamma_0} = -c_1 + c_2 + c_3 = |c_1| + |c_2| + |c_3| = \|L_{\rho}\|$$

and thus the representation is canonical. ∎

Note 4. Fix $\alpha \in [(\beta_0)^{\sigma}, 1]$. Using the above notation, we can write

$$\|L_{\rho}\| = \frac{\rho^2 - \rho(\gamma_0 - 1) - 1}{\gamma^0 - 1} \quad \rho \in [-\alpha^{1/\sigma}, 0].$$

Recall that $\|L_{\rho}\| = \|L_{-\alpha^{1/\sigma}}\| = 1$. Since $\|L_{\rho}\|$ is a continuous function of ρ (the selection of γ_0 is continuous in ρ), we choose $\rho^* \in [-\alpha^{1/\sigma}, 0]$ such that

$$\|L^{\rho^*}\| = \max_{\rho \in [-\alpha^{1/\sigma}, 0]} \|L_{\rho}\|$$

and let $N(\alpha) = \|L_{\rho^*}\|$

LEMMA 2.10. *$N(\alpha)$ is a continuous function of α.*

Proof. We claim $\lim_{\alpha \to \alpha_0} N(\alpha) = N(\alpha_0)$. We will show that $\lim_{\alpha \to \alpha_0^-} N(\alpha) = N(\alpha_0)$; the similar statement using right-hand limits will then follows by an identical argument. Thus fix $\alpha_0 \in [(\beta_0)^{\sigma}, 1]$ and let $\alpha_n \to \alpha_0^-$. Without loss, we may assume $\{\alpha_n\}$ is an increasing sequence, i.e., we assume $\alpha_n \leqslant \alpha_{n+1}$. Now, for $\alpha \in [(\beta_0)^{\sigma}, 1]$, we define the following function on $[-1, 0]$:

$$N_{\alpha}(\rho) = \begin{cases} \|L_{\rho}\| & -\alpha^{1/\sigma} \leqslant \rho \leqslant 0 \\ 1 & -1 \leqslant \rho < -\alpha^{1/\sigma} \end{cases};$$

thus $N(\alpha) = \max_{\rho \in [-1,0]} N_{\alpha}(\rho)$. Clearly, if the sequence of functions $N_{\alpha_n}(\rho)$ converges uniformly on $[-1, 0]$ to $N_{\alpha_0}(\rho)$ then

$$\lim_{n\to\infty} \max_{\rho \in [-1,0]} N_{\alpha_n}(\rho) = \max_{\rho \in [-1,0]} N_{\alpha_0}(\rho)$$

and we will have

$$\lim_{a_n \to a_0^-} N(\alpha_n) = N(\alpha_0).$$

Thus we now establish the uniform convergence of $N_{\alpha_n}(\rho)$ to $N_{\alpha 0}(\rho)$ by appealling to Dini's Theorem. Indeed, for fixed n, $N_{\alpha_n}(\rho)$ is continuous in ρ, as is $N_{\alpha_0}(\rho)$. Furthermore we claim that $N_{\alpha_n}(\rho)$ converges pointwise on $[-1, 0]$ to $N_{\alpha 0}(\rho)$. Pointwise convergence is clear for a fixed $\rho \leqslant -\alpha_0^{1/\sigma}$. For fixed $\rho > -\alpha_0^{1/\sigma}$, we note equation (7). Specifically note that for fixed $\rho > -\alpha_0^{1/\sigma}$, γ_0 (the zero of the numerator in (7) located in the fixed interval $[-\rho, 0]$) varies continuously with α. Thus pointwise convergence follows. Finally, we now show that

$$N_{\alpha_n}(\rho) \geqslant N_{\alpha_{n+1}}(\rho) \qquad \forall \rho \in [-1, 0]. \tag{9}$$

Since (9) is clear for $\rho \leqslant \alpha_0^{1/\sigma}$, we fix $\rho \in [-\alpha_0^{1/\sigma}, 0]$ and recall $\alpha_n \leqslant \alpha_{n+1}$. Recall also that γ_0 is the unique solution to

$$(\rho^2 - 1)\, \gamma\, |\gamma|^\sigma + \rho(1-\alpha)\, \gamma^2 + (1-\rho^2)\, \gamma - \rho + \alpha\rho = 0$$

on $[\rho, 0]$. Let γ_{0_α} denote the solution to the above for a given α. Then, rewriting the above as

$$1 - \alpha = \kappa\lambda\, |\gamma_\alpha| \left(\frac{1 - |\gamma_\alpha|^\sigma}{1 - \gamma_\alpha^2}\right)$$

with κ a positive constant (depending only on ρ), we claim that if $\alpha_n < \alpha_{n+1}$ then $|\gamma_{0_{\alpha_n}}| \geqslant |\gamma_{0_{\alpha_{n+1}}}|$. Indeed, by considering the function

$$f(x) = x \left(\frac{1 - x^\sigma)}{1 - x^2}\right)$$

defined on $[0, 1]$, where we define $f(1) = \sigma/2$, it is easy to check that f is monotone increasing on $[0, 1]$. And thus if $f(x_1) \leqslant f(x_2)$ then we must have $x_1 \leqslant x_2$. Therefore, if $\alpha_n \leqslant \alpha_{n+1}$ then $(1 - \alpha_n) \geqslant (1 - \alpha_{n+1})$ and thus $|\gamma_{0_{\alpha_n}}| \geqslant |\gamma_{0_{\alpha_{n+1}}}|$. Now since

$$N_{\alpha_n}(\rho) = \frac{\rho^2 - \rho(\gamma_{0_{\alpha_n}} - 1) - 1}{\gamma_{0_{\alpha_n}} - 1} = \frac{1 - \rho^2}{1 - \gamma_{0_{\alpha_n}}} - \rho$$

and $|\gamma_{0_{\alpha_n}}| \geqslant |\gamma_{0_{\alpha_{n+1}}}|$, it follows that

$$N_{\alpha_n}(\rho) \geqslant N_{\alpha_{n+1}}(\rho).$$

Therefore N_{α_n} converges uniformily to N_{α_0} and we conclude

$$\lim_{a_n \to a_0^-} N(\alpha_n) = N(\alpha_0).$$

A similar argument shows

$$\lim_{a_n \to a_0^+} N(\alpha_n) = N(\alpha_0)$$

and thus $N(\alpha)$ is a continuous function of α. ▮

COROLLARY 2.1. *For* $\alpha \in [(\beta_0)^\sigma, 1]$ *we have*

$$\|P_\alpha\| = \max(\|L_{-1}\|, \|L_{\rho*}\|).$$

Furthermore, $\|P_\alpha\|$ *is a continuous function of* α.

Proof. This follows from Lemma 2.10 and Lemma 2.8. ▮

LEMMA 2.11. *There exists* $\hat{\alpha} \in [(\beta_0)^\sigma, 1]$ *such that* $\|L_{-1}\| = \|L_{\rho*}\| = \|P_{\hat{\alpha}}\|$.

Proof. Recall that $\rho*$ depends only on α. For $\alpha = 1$, recall $L_{-1} = \delta_{-1} \circ P_1 = \delta_{-1}$ and thus $\|L_{-1}\| = 1$. We claim $\|L_{\rho*}\| > 1$. Indeed, using Note 4 above we have

$$\|L_\rho\| = \frac{\rho^2 - \gamma_0 \rho + \rho - 1}{\gamma_0 - 1}, \qquad \rho \in [-1, 0].$$

Furthermore, in the proof of Lemma 2.9, we see that for $\alpha = 1$, $\gamma_0 = 0$ for all ρ. Therefore,

$$\|L_{\rho*}\| = \max_{\rho \in [-1,0]} -\rho^2 - \rho + 1 > 1$$

and the claim is established. For $\alpha = (\beta_0)^\sigma$, we show $\|L_{-1}\| > \|L_{\rho*}\|$ and by Corollary 2.1 we will be done. Recall

$$\hat{T}_{\sigma+1}(t) = \left(\frac{1}{\sigma(\beta_0)^{\sigma+1}}\right) t\,|t|^\sigma - \left(\frac{\sigma+1}{\sigma\beta_0}\right) t$$

is the analog of the third degree Chebyshev polynomial T_3, where β_0 satisfies $H(\beta) = 0$. Consider

$$L_{-1}(\hat{T}_{\sigma+1}) = -\left(\frac{1}{\sigma\beta_0} - \frac{\sigma+1}{\sigma\beta_0}\right) = \frac{1}{\beta_0} > 1.$$

Therefore $\|L_{-1}\| \geqslant 1/\beta_0$ and we now show

$$\|L_\rho\| = \frac{\rho^2 - \gamma_0 \rho + \rho - 1}{\gamma_0 - 1} < \frac{1}{\beta_0} \tag{10}$$

for $\rho \in [-\beta_0, 0]$ (recall $-\alpha^{1/\sigma} = -\beta_0$) and γ_0 as in Lemma 2.9. Recall for $\rho = 0$ or $-\alpha^{1/\sigma}$, we find that $\|L_\rho\| = 1$ (since it is a point evaluation). For all other ρ, we have $\gamma_0 > \rho$ and this case is now considered. Since

$$\gamma_0 > \rho \Rightarrow (\rho^2 - 1) - \rho(\gamma_0 - 1) > (\rho^2 - 1) - \rho(\rho - 1),$$

we have

$$\frac{\rho^2 - \gamma_0 \rho + \rho - 1}{\gamma_0 - 1} = \frac{(\rho^2 - 1) - \rho(\gamma_0 - 1)}{\gamma_0 - 1} < \frac{(\rho^2 - 1) - \rho(\rho - 1)}{\gamma_0 - 1} = \frac{1 - \rho}{1 - \gamma_0}.$$

So to show (10), we prove $(1-\rho)/(1-\gamma_0) \leqslant 1/\beta_0$, or equivalently:

$$\frac{1 - \gamma_0}{1 - \rho} \geqslant \beta_0. \tag{11}$$

For fixed ρ, we use the numerator of (7) to define

$$M(\gamma) = (\rho^2 - 1)\,\gamma\,|\gamma|^\sigma + \rho(1-\alpha)\,\gamma^2 + (1-\rho^2)\,\gamma - \rho + \alpha\rho.$$

One can verify that $M(\rho\,|\rho|^{1/\sigma}) > 0$. Recalling that $M(\gamma)$ has a unique zero on $[\rho, 0]$ (with $M(\rho) < 0$ and $M(0) > 0$), we can conclude

$$\rho\,|\rho|^{1/\sigma} > \gamma_0. \tag{12}$$

Thus

$$\frac{1 - \gamma_0}{1 - \rho} > \frac{1 - \rho\,|\rho|^{1/\sigma}}{1 - \rho}$$

and so we show

$$\frac{1 - \rho\,|\rho|^{1/\sigma}}{1 - \rho} \geqslant \beta_0, \qquad \rho \in (-\beta_0, 0).$$

This inequality is clearly true at the endpoints of the interval. So set $f(\rho) = (1 - \rho\,|\rho|^{1/\sigma})/(1-\rho)$ and consider

$$f'(\rho) = 0 \Leftrightarrow -|\rho|^{1/\sigma}\left[1 + \frac{1}{\sigma} - \frac{\rho}{\sigma}\right] + 1 = 0 \Leftrightarrow |\rho|^{1/\sigma} = \frac{\sigma}{\sigma + 1 - \rho}. \tag{14}$$

We want to show that there exists a unique point in $(-1, 0)$, ρ_0, such that the last equality in (14) holds. Since $f'(0) > 0$, $f'(-1) < 0$, and

$$f''(\rho) = \frac{|\rho|^{1/\sigma}}{\sigma}\left[1 + \frac{1}{|\rho|}\left(1 + \frac{1-\rho}{\sigma}\right)\right] > 0$$

we can conclude that f has a unique minimum on $[-1, 0]$. Thus, we let ρ_0 be the unique minimum, or equivalently, the unique point satisfying $|\rho|^{1/\sigma} = \sigma/(\sigma + 1 - \rho)$. So, to accomplish (13) it remains only to show $f(\rho_0) > \beta_0$. Using the last equality in (14) we have

$$f(\rho_0) = \frac{1 - \rho_0\,|\rho_0|^{1/\sigma}}{1 - \rho_0} = \frac{1 - \rho_0(\sigma/(\sigma + 1 - \rho_0))}{1 - \rho_0} = \frac{\sigma + 1}{\sigma + 1 - \rho_0}.$$

Since $\rho_0 \in (-1, 0)$ we have $(\sigma + 1)/(\sigma + 1 - \rho_0) \geqslant (\sigma + 1)/(\sigma + 2)$. We claim $(\sigma + 1)/(\sigma + 2) \geqslant \beta_0$. This is equivalent to showing $H((\sigma + 1)/(\sigma + 2)) > 0$.

$$H\left(\frac{\sigma+1}{\sigma+2}\right) > 0 \Leftrightarrow \sigma\left(\frac{\sigma+1}{\sigma+2}\right)^{\sigma+1} + (\sigma+1)\left(\frac{\sigma+1}{\sigma+2}\right)^{\sigma} - 1 > 0$$

$$\Leftrightarrow 2(\sigma+1)^{\sigma+2} - (\sigma+2)^{\sigma+1} > 0 \Leftrightarrow 2x^{x+1} > (x+1)^x$$

for $x \geqslant 2$ where $x = \sigma + 1$. A simple calculus argument using logarithms shows this to be true. Thus (13) implies (11) and this allows us to conclude (10). The continuity of $\|L_{-1}\|$ and $\|L_{\rho^\star}\|$ with respect to α completes the proof of Lemma 2.11. ∎

Proof of Theorem 2.1. Let $\hat{\alpha}$ be as in Lemma 2.11. We now construct the $E_{P_{\hat{\alpha}}}$ operator that will leave V invariant. We define

$$E_{P_{\hat{\alpha}}} = \sum_{i=1}^{4} (x_i \otimes y_i)\,\lambda_i,$$

where $\{(x_i, y_i)\}$ are the following extremal pairs and λ_i's are defined below. Let $x_1 \in \mathscr{F}_\beta$ such that $\|L_{-1}\| = L_{-1}x_1$ and $y_1 = \delta_{-1}$. (x_1, y_1) is clearly an extremal pair for $P_{\hat{\alpha}}$. By the symmetry of $P_{\hat{\alpha}}$, we next define $x_2 = x_1^*$ (recall $x^*(t) = x(-t)$) and $y_2 = \delta_1$ as the second pair. Finally, let $x_3 \in \mathscr{F}_\gamma$ be such that $\|L_{\rho^\star}\| = L_{\rho^\star}x_3$ and $y_3 = \delta_{\rho^\star}$; the fourth extremal pair will be $x_4 = x_3^*$ and $y_4 = \delta_{-\rho^\star}$. Now setting $\lambda_1 = \lambda_2 = \lambda$ and $\lambda_3 = \lambda_4 = (1 - \lambda)$, we write

$$E_{P_{\hat{\alpha}}} = \lambda[(x_1 \otimes y_1) + (x_2 \otimes y_2)] + (1 - \lambda)[(x_3 \otimes y_3) + (x_4 \otimes y_4)]$$

for some $\lambda \in (0, 1)$.

Note that

$$E_{P_{\hat{\alpha}}}(1) = \lambda[x_1 + x_2] + (1 - \lambda)[x_3 + x_4] \in \Pi_2$$

since the odd terms vanish. Similarly

$$E_{P_{\hat{\alpha}}}(t^2) = \lambda[x_1 + x_2] + (1-\lambda)(\rho^*)^2 [x_3 + x_4] \in \Pi_2.$$

Writing

$$x_1(t) = At\,|t|^\sigma + Bt^2 + Ct + D \qquad \text{and} \qquad x_3 = at\,|t|^\sigma + bt + ct + d$$

we have

$$\begin{aligned} E_{P_{\hat{\alpha}}}(t) &= \lambda[-x_1 + x_2] + (1-\lambda)\,\rho^*[x_3 - x_4] \\ &= 2[\lambda(At\,|t|^\sigma + Ct) + (1-\lambda)\,\rho^*(at\,|t|^\sigma + ct)]. \end{aligned}$$

We want to choose λ such that

$$\lambda(-A - \rho^* a) + \rho^* a = 0.$$

So setting

$$\lambda = \frac{\rho^* a}{A + \rho^* a}$$

and checking back to the definitions of the coefficients of $\mathscr{F}_\beta$ and $\mathscr{F}_\gamma$ elements, we find $a < 0$ and $A > 0$. Therefore

$$0 < \lambda < 1$$

and $P_{\hat{\alpha}}$ is minimal. ∎

The following table lists some norms of minimal projections from different overspaces.

σ	$\hat{\alpha}$	$\|P_{\hat{\alpha}}\|$
1	0.92918	1.21584
2	0.88571	1.19918
3	0.85287	1.18680
5	0.80002	1.16810
8	0.73733	1.14792
10	0.70328	1.13776
11	0.68802	1.13337

References

1. B. L. Chalmers and F. T. Metcalf, Determination of a minimal projection from $C[-1, 1]$ on the quadratics, *Numer. Funct. Anal. Optim.* **11** (1990), 1–10.

2. B. L. Chalmers and F. T. Metcalf, A characterization and equations for minimal projections and extensions, *J. Oper. Theory* **32** (1994), 31–46.
3. G. J. O. Jameson, A lower bound for the projection constant of P_2, *J. Approx. Theory* **51** (1987), 163–167.
4. T. J. Rivlin, "Chebyshev Polynomials," pp. 97–99, New York, 1990.